An “Introduction to Calculus” Course Based on DEVS: Implications of a Discrete Reformulation of Mathematical Continuity

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Abstract. Traditional teaching of the calculus rests on the firm foundations established in the pre-computer age by brilliant mathematicians. Despite marvelous advances in computer-based support for calculus instruction, such non-intuitive and non-algorithmic definitions render the teaching of the calculus intrinsically difficult. The success of discrete event methods for simulating large continuous models suggests the applicability of computationally-based discrete methods to modeling of continuous phenomena and more fundamentally, for characterizing real-world continuity We present a concept of event set and an associated measure of activity that are able to capture the underlying intuition of continuity as well as to provide a direct measure of the computational work needed to represent continuity on a digital computer. This suggests the possibility of dispensing with the mysteries of traditional calculus to teach an equivalent calculus based on finitary concepts of event sets and their straightforward algorithmic manipulation.

Traditional teaching of the calculus rests firmly on the foundations established at the turn of the last century. Likewise, developers of today’s marvelous technological support tools – graphing calculators, high speed computers, math analysis packages, and the like – continue to work with the definitions of continuity as formulated in the pre-computer age. Developers are focused on increasing the power of digital automata to work with the artifacts of the “received” calculus such as derivatives and integrals, whose basic definitions remain unquestioned. While this approach yields increasingly powerful math packages such as Matlab and Mathematica, it closes off development to potentially new approaches to representing continuous aspects of the real world and to more efficient computational means to model and simulate such phenomena. As we shall argue, it may also render the teaching of the calculus intrinsically difficult and vulnerable to being supplanted by more effective modern approaches.

According to (Devlin, 2000) it is no wonder that students have extreme difficulty grasping the traditional Cauchy-Weierstrass definition of continuity since it fails to capture their intuitions about continuous change in the real world. From a modeling perspective, continuity is not an inherent property of the real world. Nevertheless, we seem to have strong perceptual and psychological propensities connect a series of discrete events in time and space with a smooth, gapless mental line. From an evolutionary biology perspective, such imagined continuity well be an ingrained illusion that evolved to enable prediction of motion of perceived objects whether threatening or full of meal-time prospects. In any event, the accepted mathematical definition of continuous function, while capturing some of our intuitions, also allows well-known “pathological” cases that suggest it is not a good model of what we intuit as continuous phenomena. More particularly the standard epsilon-delta definition allows continuous curves that lack derivatives at some, or even, all of their points, a fact that seems to amaze most of us. Perhaps we tend to see continuity as relatively slow change and hence as also smoothly predictable at least in the short term, i.e., the

--1. Pre-calculus
--2. Limits and Continuity
--3. The Derivative
--4. Techniques and Theory of Differentiation
--5. Applications of the Derivative
--6. Integration
--7. Applications of Integration
--8. Transcendental Functions
--9. Methods of Integration
--10. Geometry, Curves and Polar Coordinates

Figure 1 Traditional Introduction to Calculus1 -- Adapted from, Calculus on the Web, cow.math.temple.edu intuition of continuity includes existence of forward projecting derivatives. The mismatch with intuition may be at the root of the difficulties that most students find in learning, and more importantly, applying the calculus. It may also explain the deep chasm that separates current formulations of continuous, from discrete, mathematics.

Today's Calculus and Its Instruction

Let's review how the calculus is currently taught before going on to suggest alternatives. Despite numerous attempts, to modernize it, make it more palatable, graphically enhance it, and dumb it down, the content of introductory courses remains pretty much as depicted in Figure 1. Key elements are the definitions of limits, continuity, the derivative and the integral. Nowadays, much time is spent on drills with graphing calculators that help compute derivatives and integrals, although much time is still devoted to analytical solution methods for symbolic differentiation and integration. The epsilon-delta definition of continuity is the cornerstone of the approach. It defines a limiting process, in a form requiring first order predicate logic to state (Vaananen 2004):

$$[\lim_{x\to a} f(x) = L] \Leftrightarrow [(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x, |x-a|<\delta \Rightarrow |f(x) - L| < \varepsilon]$$

It goes on to define continuity of a function f as holding at a given point a if

$f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$.

Finally, f is said to be continuous in an interval if it is continuous at each point in the interval.

The basic problem with the epsilon-delta definition of continuous function is that it presumes, at the start, a mapping from a continuous domain to a continuous range – requiring us to deal with not only the real numbers (Cantor cardinality □) but the set of all mappings from the □ to □ (Cantor cardinality $□^{□}$) as well. Let's look at the problem from the point of view of a procedure to determine whether an arbitrary mapping $f : □ \to □$ is continuous in an interval of its domain. To make this test requires that we check continuity at all (infinitely many) of its points. To determine continuity at a point requires taking limits, i.e., infinite sequences of points converging in both directions to the point in question.

To decide whether a function is differentiable requires yet more limit tests: A continuous function, f is has a derivative given point a if $\lim_{h\to 0} |f(x+h) - f(x)|/h$ exists. f is differentiable if it has derivatives at each point in its domain. Thus, the following procedure summarizes testing for differentiability:

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is f continuous?
      (check the epsilon-delta condition at each point in its domain)
if no, exit with "f is not continuous"
otherwise, is f differentiable?
      (check the epsilon-delta definition for
      its derivative at each point in its domain).
if no, exit with "f is continuous but not differentiable"
otherwise, exit with "f is smooth
                                        (i.e.,   continuous   and
                                        differentiable)"
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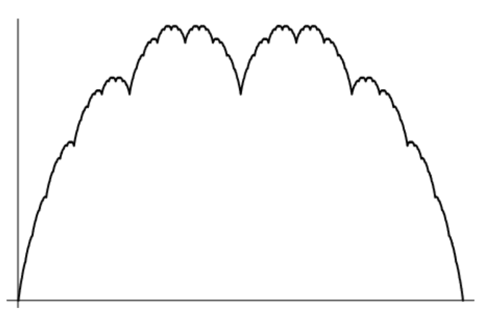

Figure 2. Blancmange Function, it is continuous, but nowhere differentiable – taken from integrals.wolfram.com/

The computational requirements of this procedure are mind-boggling. The epsilon-delta requires testing all positive real values of delta, at a point, then doing this for each point in a continuum. Further, testing for a derivative requires doing the epsilon-delta test for all positive real values of the excursion, h, again doing this for each point in a continuum.

Of course, students are never given arbitrary functions to which to apply this procedure. Indeed, they work symbolically and numerically with a class of examples drawn from well-understood elementary functions, such as polynomials, trigonometric functions, and logarithmic/exponential functions and their compositions.

Known "pathological" constructions, such as that in Figure 2, provide examples of continuous functions that are **nowhere** differentiable. They tell us not only that continuity does not guarantee differentiability, but also that differentiability at a point does not necessarily guarantee differentiability at neighboring points. Taken together, this means that the procedure for testing continuity/differentiability is not algorithmic in general, although, it can be made so for sub-classes of elementary functions.

In modeling and simulation, however, we should be fundamentally prepared to deal with non-analytical, yet still, continuous, phenomena. The general question is then: how then can we formulate continuity in such a way that algorithmic and practical approaches to detecting its presence or absence are possible? More deeply, can we remove the dependence of differential equation models on the Cauchy-Weierstrass definition and re-ground them on a rigorous and algorithmic replacement? More generally, can we stimulate development of new kinds of continuous model structures? As for education, how can we employ such a "constructivist" approach to teach a version of the calculus that is truly useable by learners? One that does not require, in principle, non-algorithmic computations— neither executable by digital computers nor by high-school juniors.

Overview of an Alternative Approach

We have seen that the classical definition starts with a complete mapping f already given. However, this situation never actually pertains in real world circumstances where we start with sensed, measured and collected raw data. Furthermore, in numerical solution of differential equations, we typically start generate finite set of events which characterizes the behavior of a system over an interval. Starting with an initial step size, we perform successive simulation runs with smaller step sizes, generating more events, until some termination decision kicks in. A similar situation holds for recent discrete event, or quantization-based, simulations, except that the quantum size is the controlled parameter. In both cases, the output for each run is a series of time-stamped events that presumably approximate the continuous curves of the model. It is natural therefore, to take a finite set of events as the working data set for any consideration of continuity.

Starting from a finite set of events we consider a refinement process that adds more events on demand. We are interested in conditions where after some reasonably few steps, we can terminate the refinement process without sacrificing essential qualitative information that might be acquired were we to continue. Consider the following as a paradigmatic example: we use a regular polygon with n sides and compute the ratio of its perimeter to the distance from any vertex to its center. Let's do this for $n = 2,3,...$ after some small n, say 10, we notice that the successive differences in the computed ratio get small enough that we are willing to stop – we are willing to gamble that the ratios to be encountered if we persist will differ from the last one obtained by less than the smallest difference observed so far. Of course, in this example, there is an analytical proof that the ratio will converge to 2π so there is no gambling involved. However, imagine doing the same process for an arbitrary planar figure where no such proof of convergence exists. We are making the claim that, in practice, working with continuous functions is governed by a process of refinement, in which we expect that after some initial data have been gathered, the information we are looking for can be accepted to some level of desired residual uncertainty. Practically speaking therefore, the adjective"continuous" is applied to situations in which there are no surprises to be expected after some point in the refinement process.

In a word, continuity as currently formulated is a Platonic ideal resting on solid, but fundamentally non-algorithmic, foundations. But there must be properties of continuous functions that make them practical to work with, otherwise, they wouldn't dominate the tool chests of scientists and engineers? What are such properties? More specifically, what information stabilizes and what incremental properties govern a refinement process whose data set is sampled from a continuous function? We would like these properties to be quite intuitive and lend themselves to algorithmic and efficient computations. More particularly, we would like the approach to be implementable in the discrete event simulation paradigm, as formalized by DEVS (Zeigler et. al, 2000).

Refinement

Let's imagine that we are drawing successive sets of samples from a continuous curve that is defined on a finite interval as illustrated in Figure 3. Samples, i.e., pairs $(t, f(t))$, are events and refinement means that successive sets of events are obtained by adding new samples without ever dropping old ones. We start with the intuition that as we add samples we expect that nearby domain points, t will have nearby values or images, $f(t)$ – the essence of intuitive continuity. If the values jump around too much – if there is too much "noise" – we are reluctant to accept that the underlying curve is continuous. As we continue to fill in the interval, eventually domain points will get closer and closer and we expect that their values will likewise get closer and closer.

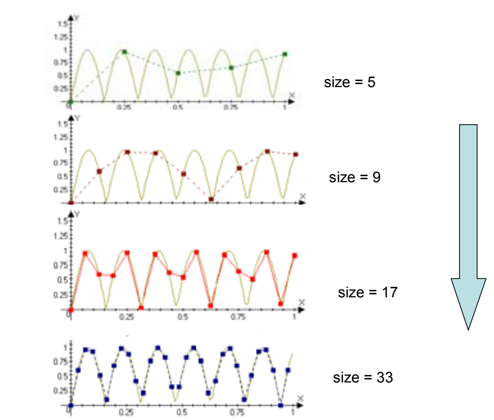

Figure 3 Event set refinement sequence that samples a rectified sine wave

So let us do the following: define a measure of variation for an event set. We expect that the value of this measure will increase as we add samples. But what is critical is that once we have enough to get the qualitative characteristics of the curve, the measure will stop growing. Indeed, if we are lucky enough to start with the right samples then the measure should stay constant from the very start of the refinement process.

The measure we show to have these properties is the sum of magnitudes of successive differences of sample values. By qualitative characteristics of the curve we mean its *form factor* defined to be the locations and values of its minima and maxima. We restrict the curves of interest to those for which there are only a finite number of such extreme points in the finite interval of interest.[1] We can show that for such a curve, if the initial sample set includes all extrema, then the variation measure must remain constant as we add in new samples. If the sample set does not include these points, then the measure will grow rapidly until points are included that are close enough to the extrema so both the sum and the form factor information will stabilize. This convergence is illustrated in Figure 4 for the refinement sequence in Figure 3.

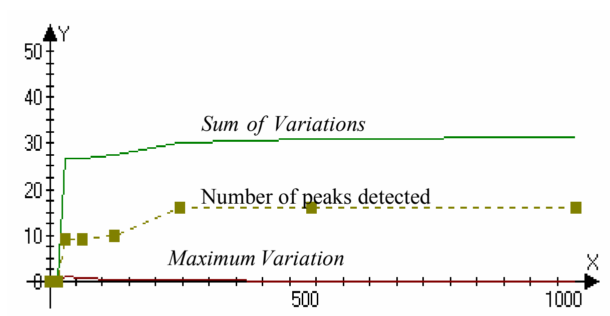

Figure 4. Convergence of the Sum and Maximum variation measures. Also shown is the convergence to the form factor.

The sum measure will converge for bounded piecewise continuous curves (with finite jumps) as well. To detect such curves, we can monitor the maximum of the successive difference magnitudes. This measure will decrease to zero for continuous curves, as illustrated in Figure 4, and to non-zero values for piecewise continuous curves (that are continuous except for a finite number of discontinuities.). More details and results that confirm agreement with intuition are presented in (Zeigler 2004).

[1] More generally, we require that there be a finite number of extrema in any finite sub interval if we allow the function to be defined over an infinite interval.

The interesting thing about the sum measure is that it is actually the activity measure for DEVS simulation of continuous models (Jammalamadaka ,2003;Akerkar 2004). The activity number divided by a quantum gives the smallest number of threshold crossings that are needed to simulate the behavior of the model using that quantum size. The number of transitions and hence the execution time are directly related to the threshold crossings. Thus we conclude that the activity measure is a fundamental characteristic of continuous phenomena that both captures the underlying intuition of continuity as well as providing a direct measure of the computational work needed to represent continuity on a digital computer.

--1. Pre-calculus
--2. Event Sets
--3. Basic Event Set Types
--4. Integrals and Derivatives of Event Sets
--5. Event Set Representation of Continuous Functions
--6. Event Set Differential Equations

Figure 5 DEVS-based Introduction to Calculus

In sum, this formulation offers a new way to unify the computational representation of both continuous and discrete phenomena and simulate them with the greater efficiency and flexibility afforded by object-oriented discrete event environments. Our interest in this paper is on the implications for improving the teaching of calculus to which we now turn.

Introduction to Calculus Based on DEVS

Originally developed for application to simulation of ordinary, and later, partial differential equations, the activity concept has become the starting point for a more general and fundamental formulation. The formulation is motivated by a desire to reconcile everyday discrete digital computation with the higher order continuum of traditional calculus. We believe that it succeeds in reducing the traditional calculus, at least in the basic form needed for an introduction, essentially to computer science without need for advanced analytical mathematics. A major application therefore is to the revamping education in the calculus to dispense with its mysterious tenets that are too difficult to convey to learners.

Figure 5 outlines a course in calculus that would be the basis for training students in modeling and simulation. The pre-calculus requirements in this approach would be much more concentrated on fundamentals of set theory as suggested in Figure 6. On the basis of such background, the concepts of event sets would be introduced and the two basic event set types, domain-based and range-based would be characterized and compared. Note that these topics would precede any formal discussion of continuity. The emphasis is on what constitutes data in modeling and simulation, namely sets of events that are indexed by time and space. Domain-based event sets are characterized by having equally spaced domain points – for example, as gathered by fixed time stepped simulation (in which time is the domain) or by equal cell sizes spanning a spatial representation. Range-based events are characterized by having equally spaced range values. In quantization-based simulations, events are generated by crossing of thresholds that are spaced apart from each other by a

--1. pre-calculus
----1. Sets
----2. Relations
----3. Functions
----4. Examples
--2. **Event Sets**
----1. Definitions
----2. Measures of Variation
----3. Extrema
----4. Refinement
--3. **Basic Event Set Types**
----1. Domain-based Event Sets
----2. Range-Based Event Sets
----3. Mapping between Types
----4. Ratio of Sizes

--4. Integrals and Derivatives of Event Sets
----1. Integral of an Event Set
----2. Derivative of an Event Set
----3. Integral of Within-the-box Refinement
----4. Fundamental Theorem Relating Integrals to Derivatives

--5. Event Set Representation of Continuous Functions
----1. Centrality of Within-the-box Refinement
----2. Uncertainty Metric
----3. The Range-Based Representation Can Be Arbitrarily More Efficient Than Its Domain-Based Equivalent
----4. Convergence of Representations
----5. Multi-dimensional Representations

--6. Event Set Differential Equations
----1. Posing the Problem
----2. Solution in One-Dimension
----3. Comparison with Traditional Differential Equations

Figure 6 Topics Covered in DEVS-based Introduction to Calculus quantum. Such an event set has values (thresholds) differing by one quantum with times that are not necessarily equally spaced. Mapping between such event set types shows how to go from one to the other and the effect on the number of events that are employed. Range based event sets are much more efficient with respect to number of samples needed, especially where there are periods of little change. The Integrals and derivatives of event sets are defined and related to each other by the analog of the Fundamental theorem of calculus. In contrast to the deep nature, and advanced proof mechanisms required for the latter, the inverse relation between integration and differentiation is immediately apparent in the event set formulation.

Key to this development is the refinement process mentioned earlier and a particular form of refinement called within-the-box refinement. This refinement matches what is generated by sampling from a continuous function. However, it is a very simple rule that does not require anything like the complexity of the calculus to understand. Likewise, the concepts of extrema (minima and maxima) are developed without reference to continuity. Furthermore, straightforward algorithms to generate the form factor of an event set are presented. The form factor, as discussed earlier, leads to a unique decomposition of an event set into a sequence of monosets, i.e., event sets whose events are either non-increasing or non-decreasing in value. Indeed, the monosets are easily seen to alternate between the non-increasing and the non-decreasing types.

Finally, with this background, the event set representation of continuous functions is presented. This statement should be clarified – we show that within-the-box refinement sequences preserve the form factor as refinement proceeds. The basic properties of continuity include the intermediate value property and the mean value property, and the relationship between integration and differentiation. There are as usually expressed as unproved theorems in calculus textbooks. In contrast, in the proposed approach, they are seen to be consequences of the preservation of the form factor by within-the-box refinement, the monoset decomposition it enables and the monoset characteristics of the integral and derivative forms. Carrying this one step further, we can formulate the event set version of differential equations. Interestingly, it becomes clear that in one dimension, the range-based event set is the preferred form that allows us to write an explicit solution to the equation (in the same way that integration by separation of variables enables a general analytical solution in the traditional approach). Further, we show how the range-based representation of continuous functions can result in considerable efficiency of representation compare to that of its domain-based counterpart for the same uncertainty (as discussed earlier) in representation of continuous functions.

Conclusions

We have presented a concept of event set and an associated measure of activity that are able to capture the underlying intuition of continuity as well as to provide a direct measure of the computational work needed to represent continuity on a digital computer. We believe that this approach opens up the possibility of dispensing with the mysteries of traditional calculus to teach an equivalent calculus based on finitary concepts of event sets and their straightforward algorithmic manipulation. We have presented the outlines of an introductory course in the calculus as taught on the proposed basis. Software tools have been developed to support such instruction. One next step is to try out such a course on appropriate sets of students and compare students' abilities to grasp the concepts and use them to construct models of real-world continuous phenomena with the outcomes of current instruction paradigms. Any volunteers?

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